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# ON EQUATIONS DESCRIBING THE TRANSVERSE VIBRATIONS OF ELASTIC BARS

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An equation is constructed for the transverse vibrations of isotropic elastic bars of centrally-symmetric cross section, which is a refinement of the Bernoulli-Euler equation. A comparison with other results on refined equations of bars is given.

**1. Formulation of the problem.** Let us consider a rectilinear elastic bar of length $2l$ and constant cross section $\Omega$ in a Cartesian $x$, $x^\alpha$ coordinate system. (The Greek indices take on the values 1, 2). In the undeformed state the bar axis coincides with the $x$-axis, and the center of gravity with the origin. Let $w$, $w_\alpha$ denote the projections of the displacement vector on the $x$, $x^\alpha$ axes, and $h$ the bar diameter (the maximum distance between points of the boundary of the domain $\Omega$). The bar is loaded along the lateral surface and along the endfaces by surface forces with components $p$, $p_\alpha$ dependent on the coordinates and the time $t$. There are no mass forces.

Let us limit ourselves to the examination of bars of centrally-symmetric cross section. A section which contains a point with coordinates $-x^\alpha$ together with a point with coordinates $x^\alpha$ is understood to be centrally-symmetric. In this case the general dynamic problem decomposes into two independent problems, namely, the transverse vibrations ($w$ is odd, $w_\alpha$ are even functions of $x^\alpha$), and the longitudinal vibrations and torsion ($w$ is even and $w_\alpha$ are odd functions of $x^\alpha$, see Appendix). The problem of transverse vibrations will henceforth be considered.

Saint Venant [1] solved the static problem of the transverse deformation of a bar, whose lateral surface is load-free. The solution in terms of displacements is

$$w = e_\alpha(x)x^\alpha + g(x^\alpha), \quad w_\alpha = u_\alpha(x) + {}^1/_4\chi_\beta(x)R_\alpha^\beta \tag{1.1}$$

where $e_\alpha$, $u_\alpha$, $\chi_\alpha$ are functions of $x$, $g(x^\alpha)$ satisfies the equation

$$\mu\Delta g = -x^\alpha\partial_x[(\lambda+\mu)\chi_\alpha + (\lambda+2\mu)\partial_x e_\alpha]$$

at all points of the section $\Omega$ and the condition

$$\frac{\partial g}{\partial n} = -n^\alpha\left[e_\alpha + \partial_x u_\alpha + \frac{1}{4}\partial_x\chi_\beta R_\alpha^\beta\right]$$

on the boundary $\Gamma$. Here

$$I^{\alpha\beta} = \frac{1}{\Omega}\int_\Omega x^\alpha x^\beta\, d\Omega, \quad R^{\alpha\beta} = 2(x^\alpha x^\beta - I^{\alpha\beta}) - \delta^{\alpha\beta}(x^\gamma x_\gamma - I_\gamma^\gamma), \quad \partial_x = \frac{\partial}{\partial x}$$

$\Delta$ is the Laplace operator in the variables $x^\alpha$, $\lambda$ and $\mu$ are Lamé parameters and $n^\alpha$ are components of the unit external normal vector to the contour $\Gamma$.

The functions (1.1) represent an exact solution of the elasticity theory equations if the surface forces on the endfaces, statically equivalent to a given force and a couple, are selected in a special way. It has been proved by Toupin [2] that the stresses caused by a self-equilibrated load on the bar endface decrease exponentially with distance from the endface. An estimate $\gamma_0 / h \leqslant \gamma$ has been obtained in [3] for a constant $\gamma$ in the exponential, where $\gamma_0$ depends only on the shape of the cross section. Therefore, the stresses caused by a self-equilibrated load are functions of boundary-layer type, and the Saint-Venant solution describes the internal state of stress of the bar completely (the difference between the Saint-Venant and exact solution does not exceed $ch^q$ as $h \to 0$, where $q$ is any arbitrarily large number).

The problem of the state of stress of a bar under an arbitrary lateral load could be reduced to the above if some solution of the elasticity theory equations satisfying the boundary conditions on the lateral surface were to be constructed successfully. However, such solutions are obtained only in some particular cases [4 — 6]. An exact solution of the problem has been found successfully in the dynamical theory of beam bending for the vibrations of an infinite circular cylinder whose lateral surface is load-free [7, 8], and of an infinite rectangular bar in a state of plane strain or plane stress [9]. Difficulties in obtaining an exact solution make necessary the construction of approximate equations for the theory of bars.

The most consistent approach is derivation of approximate equations by asymptotic methods using the presence of the small parameter $h / l$. The papers [10, 11] are devoted to asymptotic methods in the theory of bars. Approximate equations of the static problem of deformation of a bar loaded arbitrarily along the lateral surface are obtained in [10] by the method of asymptotic integration of the equations of three-dimensional elasticity theory. This method was used in [11] to derive approximate equations of the longitudinal vibrations of circular bars. Asymptotic methods have not been applied to the problem of the transverse vibrations of beams.

The approximate equations of the theory of vibrations of bars are usually obtained by using heuristic hypotheses relative to the nature of the states of strain and stress. The concept of an asymptotically exact model arises in an appraisal of these hypotheses from the viewpoint of the asymptotic approach. Namely, we call the model (or equations) asymptotically exact if all the elastic effects whose energy is of the same order of smallness are included in the consideration when taking some effect into account. The model corresponding to the hypothesis of plane sections (the Bernoulli-Euler model) is asymptotically exact and yields the first approximation (see Sect. 7).

Extensive literature (see [12 — 17]) is devoted to constructing refinements of the Bernoulli-Euler equations. However, no asymptotically exact equations have been obtained in the papers mentioned (see Sect. 8).

The purpose of this paper is the formulation of hypotheses for the second approximation and the foundation of the asymptotic accuracy of the corresponding model.

We derive a system of equations describing the transverse vibrations of a bar from the condition of an extremum of the functional of linear elasticity theory [18]

$$I = \int_{t_1}^{t_2} dt \left[ \int_V \left( \rho \frac{v^2}{2} - U \right) dV + \int_\sigma (pw + p_\alpha w^\alpha)\, d\sigma \right] \qquad (1.2)$$

$$2U = \lambda (\varepsilon_\alpha{}^\alpha)^2 + 2\lambda \varepsilon \varepsilon_\alpha{}^\alpha + (\lambda + 2\mu)\, \varepsilon^2 + 2\mu \varepsilon_{\alpha\beta} \varepsilon^{\alpha\beta} + 4\mu \varepsilon_\alpha \varepsilon^\alpha$$

$$\varepsilon_{\alpha\beta} = w_{(\alpha,\beta)}, \quad \varepsilon = \partial_x w, \quad 2\varepsilon_\alpha = w_{,\alpha} + \partial_x w_\alpha$$

Here $V$ is the volume occupied by the bar, $\sigma$ is the surface bounding the volume $V$, $v$ is the modulus of the velocity vector of the bar particles, and $U$ is the internal energy of unit volume. A comma in the subscripts means differentiation with respect to $x^\alpha$. The variations at the initial $t_1$ and final $t_2$ times are considered to be zero.

Making a hypothesis relative to the dependence of the displacement vector on the transverse coordinates and integrating over the bar cross section in (1.2), we obtain a functional defined in terms of functions of $x$, $t$.

The integrand in the functional will contain terms of different orders of smallness in $h$. The $n$-th approximation equations are obtained in taking the variation of the functional in which terms of order $h^{2n}$ are contained and terms of higher order in $h$ are discarded. In determining the orders of magnitude we consider that differentiation with respect to $x$ and $t$ does not decrease the order of smallness.

It is known from the exact solutions of the problem of free transverse vibrations that there exists a countable number of qualitatively distinct types of vibrations, and correspondingly, of branches of the dispersion curve. The Bernoulli-Euler equations describe long-wave vibrations corresponding to the first branch of the dispersion curve. Differentiation with respect to time hence increases the order of smallness of the quantities by one. We take a more general assumption in constructing refinements to the Bernoulli-Euler equations: differentiation with respect to time does not decrease the order of smallness of the quantities. We consider the time dependence of the external forces to be such that this assumption is not violated.

To obtain the second approximation equations, we take the following hypotheses relative to the displacement vector components:

$$w = e_\alpha(x, t)\, x^\alpha + g(x, x^\alpha, t), \quad w_\alpha = u_\alpha(x, t) + {}^1/_4\, \chi_\beta(x, t)\, R_\alpha{}^\beta \qquad (1.3)$$

In fact the displacement vector is represented in the same form as in the Saint-Venant solution (1.1). We assume that the functions $u_\alpha$, $e_\alpha$, $\chi_\alpha$ are of order $h^0$, $\varphi_\alpha = e_\alpha + \partial_x u_\alpha$ is of order $h^2$ and $g$ is of order $h^3$. With respect to the external forces, we assume that $p \sim h^2$, $p_\alpha \sim h^3$ on the lateral surface, $p \sim h$, $p_\alpha \sim h^2$ on the endfaces. Henceforth, for simplicity we also assume that $p = m_\alpha(x, t)\, x^\alpha$ on the endfaces. The foundation of the hypotheses (1.3) is considered in Sect. 7.

Because of the definition of $R_\alpha{}^\beta$, the functions $u_\alpha(x, t)$ have the meaning of a mean transverse displacement $u_\alpha(x, t) = \langle w_\alpha \rangle$.

Without limiting the generality, the constraint

$$\langle g x^\alpha \rangle = 0 \qquad (1.4)$$

can be imposed on the function $g$.

**2. Second approximation equations.** After substituting (1.3) into (1.2) and discarding small terms of order $h^6$ and higher, we obtain a second approximation functional

$$2I = \rho\Omega \int_{t_1}^{t_2} dt \int_{-l}^{l} \left(\partial_t u_\alpha \partial_t u^\alpha + I^{\alpha\beta} \partial_t e_\alpha \partial_t e_\beta + \frac{1}{16} \langle R^\alpha_\gamma R^{\beta\gamma} \rangle\, \partial_t \chi_\alpha \partial_t \chi_\beta \right) dx - \qquad (2.1)$$

$$\Omega \int_{t_1}^{t_2} dt \int_{-l}^{l} \{[(\lambda + \mu)\chi_\alpha \chi_\beta + 2\lambda \chi_\alpha \partial_x e_\beta + (\lambda + 2\mu)\, \partial_x e_\alpha \partial_x e_\beta]\, I^{\alpha\beta} + \mu\Phi\}\, dx +$$

$$\int_{t_1}^{t_2} dt \int_\sigma \left(2p_\alpha u^\alpha + 2p e_\alpha x^\alpha + \frac{1}{2} p_\alpha \chi_\beta R^{\alpha\beta}\right) d\sigma$$

$$\partial_t = \frac{\partial}{\partial t}, \quad \langle A \rangle = \frac{1}{\Omega}\int\limits_{\Omega} A\, d\Omega$$

$$\Phi = \frac{1}{\Omega}\left[\int\limits_{\Omega}\left(\varphi_\alpha + \frac{1}{4}\partial_x\chi_\beta R_\alpha^\beta + g_{,\alpha}\right)^2 d\Omega - \frac{1}{\mu}\int\limits_{\Gamma} pg\, d\Gamma\right] \tag{2.2}$$

The function $g$ enters $I$ just in terms of $\Phi$. For $p = 0$ the quantity $\mu\Phi$ has the meaning of a shear energy. The required value $g$ evidently makes the functional $\Phi$ a minimum. The minimum value $\Phi_0$ of the functional $\Phi$ is a function of $\varphi_\alpha$, $\partial_x\chi_\alpha$. Setting $\Phi = \Phi_0$ in (2.1) we obtain a functional defined by functions of $x$ and $t$. Variation of this functional results in the second approximation equations and boundary conditions. To evaluate $\Phi_0$ in the functional $\Phi$ it is convenient to introduce $G$ as the required function in place of $g$ by means of the formula

$$G = g + x^\alpha\left[\varphi_\alpha + {}^1/_{12}\left(R_\alpha^\beta - 4I_\alpha^\beta + 2\delta_\alpha^\beta I_\gamma^\gamma\right)\partial_x\chi_\beta\right] \tag{2.3}$$

Condition (1.4) for the function $g$ goes over into a condition for $G$

$$\langle Gx^\alpha\rangle = \varkappa^\alpha,\ \varkappa^\alpha = I^{\alpha\beta}\varphi_\beta + T^{\alpha\beta}\partial_x\chi_\beta \tag{2.4}$$

$$T^{\alpha\beta} = {}^1/_4\left(I^{\alpha\beta}I_\gamma^\gamma + {}^1/_3 I^{\alpha\beta\gamma}{}_\gamma - 2I^{\alpha\gamma}I_\gamma^\beta\right),\ I^{\alpha\beta\gamma\nu} = \langle x^\alpha x^\beta x^\gamma x^\nu\rangle$$

Seeking the minimum of the functional $\Phi$ and $G$ according to the condition (2.4), we obtain

$$\Delta G = \gamma_\alpha x^\alpha \tag{2.5}$$

$$\frac{\partial G}{\partial n} = \frac{1}{3}\left(\delta_\alpha^\beta x^\gamma x_\gamma - x_\alpha x^\beta\right) n^\alpha \partial_x\chi_\beta + \frac{1}{\mu}p \text{ on } \Gamma$$

Here $\gamma_\alpha(x, t)$ are Lagrange multipliers corresponding to the condition (2.4). The boundary condition for the function $G$ turns out to be simpler than for $g$. The function $\Phi_0(\varkappa_\alpha, \partial_x\chi_\alpha)$ is a quadratic form of its arguments and can be represented as

$$2\Phi_0(\varkappa_\alpha, \partial_x\chi_\alpha) = A^{\alpha\beta}\partial_x\chi_\alpha\partial_x\chi_\beta + B^{\alpha\beta}\varkappa_\alpha\varkappa_\beta + 2C^{\alpha\beta}\varkappa_\alpha\partial_x\chi_\beta \tag{2.6}$$

The tensors $A^{\alpha\beta}$ and $B^{\alpha\beta}$ are symmetric, but $C^{\alpha\beta}$ is generally nonsymmetric. They all depend on the geometry of the cross section and the longitudinal load on the lateral surface $p$. These tensors can also depend on $x, t$ because of the dependence of $p$ on $x, t$. They are determined just by the cross-section geometry for a zero longitudinal load on the lateral surface.

It follows from the expression (2.2) for $\Phi$ that $\Phi_0 \sim h^4$. The coefficients in (2.6) are of the order of

$$A^{\alpha\beta} \sim h^4, \quad B^{\alpha\beta} \sim h^{-4}, \quad C^{\alpha\beta} \sim h^0$$

Varying the functional (2.1), we obtain an equation for $u_\alpha$

$$\rho\partial_t^2 u_\alpha - {}^1/_2\mu I_\alpha^\gamma\left(\partial_x B_{\gamma\beta}\varkappa^\beta + \partial_x^2 C_{\gamma\beta}\chi^\beta\right) = P_\alpha / \Omega \tag{2.7}$$

$${}^1/_2\mu I_\alpha^\gamma\left(B_{\gamma\beta}\varkappa^\beta + \partial_x C_{\gamma\beta}\chi^\beta\right) = \mp\langle p_\alpha\rangle, \quad x = \mp l$$

The equation for $e_\alpha$ is

$$\rho\partial_t^2 e_\alpha - \lambda\partial_x\chi_\alpha - (\lambda + 2\mu)\partial_x^2 e_\alpha + {}^1/_2\mu\left(B_\alpha^\beta\varkappa_\beta + \partial_x C_\alpha{}^\beta\chi_\beta\right) = N_\alpha / \Omega \tag{2.8}$$

$$I_\alpha^\beta\left(\lambda\chi_\beta + (\lambda + 2\mu)\partial_x e_\beta\right) = \mp\langle px_\alpha\rangle, \quad x = \mp l$$

The equation for $\chi_\alpha$ is

$$\rho\partial_t^2\chi_\beta \langle R_{\alpha\gamma}R^{\beta\gamma}\rangle + 16I_\alpha^\beta [(\lambda+\mu)\chi_\beta + \lambda\partial_x e_\beta] - \tag{2.9}$$

$$8\mu [\partial_x (T_{\alpha\gamma}B_\beta^\gamma + C_{\beta\alpha})\varkappa^\beta + \partial_x^2 (T_\alpha^{\ \gamma}C_{\gamma\beta} + A_{\alpha\beta})\chi^\beta] = 4D_\alpha/\Omega$$

$$\mu [(T_{\alpha\gamma}B^{\gamma\beta} + C_{\beta\alpha})\varkappa^\beta + \partial_x (T_\alpha^\gamma C_{\gamma\beta} + A_{\alpha\beta})\chi^\beta] = \mp {}^1/_2 \langle p_\beta R_\alpha^\beta\rangle \Big|_{x=\pm l}$$

Here

$$D_\alpha = \int\limits_\Gamma R_\alpha^\beta p_\beta \, d\Gamma, \quad P_\alpha = \int\limits_\Gamma p_\alpha \, d\Gamma, \quad N_\alpha = \int\limits_\Gamma p x_\alpha \, d\Gamma$$

**3. Resolving equation.** We obtain the resolving equation of the system (2.7)—(2.9) which is a refinement of the Bernoulli-Euler equation. Since the inertial and transverse forces in this latter are of identical order of smallness, and differentiation with respect to $x$ does not change the order of magnitude, it should be considered that differentiation with respect to $t$ increases the order of smallness by one. This assertion is valid if short-wave processes are excluded from consideration [19].

Solving the system (2.7)—(2.9) for $u_\alpha$, discarding the small terms $I_\alpha^\beta \partial_t^4 u_\beta$ because of the assumptions made, and considering the coefficients of the quadratic form (2.6) independent of $x$, we obtain

$$\rho\partial_t^2 u_\alpha + EI_\alpha^\beta \partial_x^4 u_\beta - \rho I_\alpha^\beta \partial_t^2 \partial_x^2 u_\beta + E\Psi_\alpha^\beta \partial_x^6 u_\beta = \frac{1}{\Omega}\left[P_\alpha + \partial_x N_\alpha + \frac{\nu}{2}\partial_x^2 D_\alpha\right] \tag{3.1}$$

where $E$ is the Young's modulus, $\nu$ is the Poisson's ratio, and $\Psi_{\alpha\beta}$ is a symmetric tensor defined in terms of the coefficients of the quadratic form (2.6)

$$\Psi_{\alpha\beta} = 4(1+\nu) B_{\alpha\beta}^{-1} + 4\nu (C_{(\alpha\gamma} B^{-1\gamma}{}_{\beta)} + T_{\alpha\beta}) + \frac{\nu^2}{1+\nu}(C_{\gamma\alpha}B^{-1\gamma\omega}C_{\omega\beta} - A_{\alpha\beta})$$

Here $B_{\alpha\beta}^{-1}$ denotes the tensor inverse to $B_{\alpha\beta}$ and $B_{\alpha\gamma}^{-1}B^{\beta\gamma} = \delta_\alpha^\beta$. Equations (3.1) are second approximation equations for the mean displacements $u_\alpha$ of transverse vibrations of a bar of centrally-symmetric cross section. In comparing the second approximation equations obtained by different authors, it should be kept in mind that mutually equivalent equations within the framework of this approximation can differ in form [19]. In fact, let us represent the resolving equation in the case of free vibrations in the operator form

$$(L_1 + h^2 L_2)u = (M_1 + h^2 M_2)P$$

where $L_1$, $M_1$ are operators corresponding to the Bernoulli-Euler equation. All equations of the form

$$(L_1 + h^2 L_2')u = (M_1 + h^2 M_2')P$$

are equivalent to the given equation if ($D$ is some differential operator)

$$L_2' - L_2 = DL_1, \quad M_2' - M_2 = DM_1$$

Selecting the operator $D$ in a suitable manner, we obtain the resolving equations equivalent to (3.1) but without the term $\partial_x^6 u_\beta$

$$\rho\partial_t^2 u_\alpha + EI_\alpha^\beta \partial_x^4 u_\beta - \rho [I_\alpha^\beta + I^{-1\beta}{}_\gamma \Psi_\alpha^\gamma] \partial_t^2 \partial_x^2 u_\beta = \frac{1}{\Omega}\left[P_\alpha + \partial_x N_\alpha + \frac{\nu}{2}\partial_x^2 D_\alpha - I^{-1\beta}{}_\gamma \Psi_\alpha^\gamma \partial_x^2 (P_\beta + \partial_x N_\beta)\right] \tag{3.2}$$

($I_{\alpha\beta}^{-1}$ is the inverse tensor to $I_{\alpha\beta}$).

**4. Coefficients of the resolving equation, separation of the vibrations and symmetry of the cross section.** The system (3.2) contains two equations for the displacement vector component $u_\alpha$. In general, the vibrations of one axis of the bar cause vibrations along the other. The question arises as to whether a coordinate system $x^\alpha$ exists in which the vibrations along two axes turn out to be independent? The answer to this question is affirmative for the first approximation equation, since the tensor $I^{\alpha\beta}$ can be reduced to diagonal form by an orthogonal transformation. In the case of a second approximation equation, the answer is related to the symmetry properties of the cross section.

The cross sections can be classified by referring to one class of sections which are invariant relative to the same subgroup of a group of orthogonal transformations on a plane. Each symmetry group will contain a rotation through the angle $\pi$ since the section is centrally-symmetric.

The tensors (4.1) are invariant relative to the symmetry group of the cross section. (This assertion is valid if the longitudinal load on the lateral surface $p$ has the symmetry of the cross section). According to the German-Hermann theorem (see [20, 21]), a tensor of second rank which is invariant relative to the group of rotations containing a rotation through an angle less than $\pi$ is spherical. Hence, $A^{\alpha\beta} = A\delta^{\alpha\beta}$, etc., for all tensors (4.1) for cross sections invariant relative to such groups. According to (3.2), the transverse vibrations are separated in any orthogonal coordinate system.

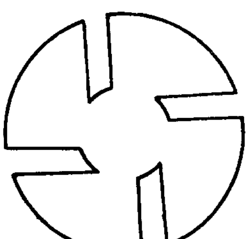

Fig. 1

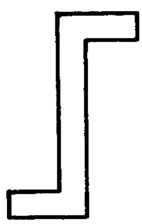

Fig. 2

Shown in Fig. 1 is a cross section which is invariant relative to rotations through $\pi / 2$. The tensors introduced above which characterize this cross section are spherical, and the corresponding bar will behave as a circular bar for transverse vibrations in a second approximation. This fact was not evident beforehand since not even two axes exist in the initial three-dimensional problem along which the transverse vibrations occur separately.

There are just two subgroups of the orthogonal group which contain no rotation through an angle less than $\pi$. One consists of two nontrivial (different from the identity) transformations: rotation through an angle $\pi$ and reflection relative to some axis $L$. In this case, by selecting the $x^1$-axis along $L$, the $x^2$-axis perpendicular to $L$, we obtain $A^{12} = A^{21} = 0$, etc. for all the tensors (4.1) from the condition of invariance of these tensors.

Therefore, the tensors (4.1) are simultaneously reduced to diagonal form in the $x^1$, $x^2$ coordinate system and according to (3.2) the vibrations along $x^1$ and $x^2$ are separated. Examples of such cross sections yield an ellipse and a rectangle.

Another subgroup contains one nontrivial transformation: rotation through the angle $\pi$. (An example of such a cross section is presented in Fig. 2). The tensors (4.1) can have a most general form. Nevertheless, even in this case coordinate systems in which the vibrations separate can exist in principle. It is easy to see that the necessary and sufficient conditions for separation of the vibrations are the equalities $I^\alpha_\beta \Psi^{\beta\gamma} = I^\gamma_\beta \Psi^{\beta\alpha}$ resulting from the theorem about simultaneous reduction of two symmetric matrices to diagonal form by an orthogonal transformation.

**5. Examples of evaluating the function $G$ and coefficients of the resolving equation.** In those cases when the Neumann problem (2.5) admits of simple analytical solutions, the coefficients $A^{\alpha\beta}$, $B^{\alpha\beta}$, $C^{\alpha\beta}$ of the quadratic form $\Phi_0$ in (2.6), and therefore, the coefficients of the second approximation resolving equation (3.2) can be calculated. Presented below are results for a circle, a circular ring, an ellipse, and a rectangle under the condition that the longitudinal force is $p = 0$ on the lateral surface.

1°. Circle of radius $r$

$$G = \frac{12}{7r^4}(3r^2 - x^\beta x_\beta)\, x^\alpha \varkappa_\alpha$$

$$A^{\alpha\beta} = \frac{r^4}{27}\delta^{\alpha\beta}, \quad B^{\alpha\beta} = \frac{2\cdot 96}{7r^4}\delta^{\alpha\beta}, \quad C^{\alpha\beta} = -\frac{2}{3}\delta^{\alpha\beta}$$

$$\Psi^{\alpha\beta} = \frac{r^4}{48}\,\frac{4\nu^2 + 12\nu + 7}{1+\nu}\,\delta^{\alpha\beta}$$

2°. Ring with radii $r_1$ and $r_2$

$$G = \frac{12}{7r_1^4 + 34r_1^2r_2^2 + 7r_2^4}\left(3\,\frac{r_1^2r_2^2}{x^\beta x_\beta} + 3(r_1^2 + r_2^2) - x^\beta x_\beta\right) x^\alpha \varkappa_\alpha$$

$$A^{\alpha\beta} = \frac{r_1^4 + r_1^2r_2^2 + r_2^4}{27}\delta^{\alpha\beta}, \qquad B^{\alpha\beta} = \frac{2\cdot 96}{7r_1^4 + 34r_1^2r_2^2 + 7r_2^4}\delta^{\alpha\beta}, \qquad C^{\alpha\beta} = -\frac{2}{3}\delta^{\alpha\beta}$$

$$\Psi^{\alpha\beta} = \frac{1}{54(1+\nu)}\left[\frac{4\nu^2+12\nu+9}{8}(7r_1^4+34r_1^2r_2^2+7r_2^4)+\nu(\nu+3)(r_1^4+r_1^2r_2^2+r_2^4)\right]\delta^{\alpha\beta}$$

3°. Ellipse $x_1^2/a^2 + x_2^2/b^2 \leqslant 1$

$$G = \frac{1}{8}\Gamma_\alpha x^\alpha x_\beta x^\beta + \frac{1}{6(3a^2+b^2)}\left\{\left[\frac{1}{4}(a^2-b^2)(3x_2^2 - x_1^2) - 3a^2(2a^2+b^2)\right]\Gamma_1 - \right.$$
$$\left. \frac{2}{3}(a^2-b^2)(3x_2^2 - x_1^2 + 3a^2)\,\partial_x\chi_1\right\} x_1 + [a \leftrightarrow b,\ 1 \leftrightarrow 2]$$

$$\Gamma_1 = \frac{1}{a^4(5a^2+2b^2)}\left[\frac{a^2}{3}(b^4 + 4a^2b^2 - 5a^4)\,\partial_x\chi_1 - 24(3a^2+b^2)\varkappa_1\right]$$

$$A_{11} = \frac{b^4(81a^4 + 30a^2b^2 + b^4)}{108(3a^2+b^2)(5a^2+2b^2)}, \quad B_{11} = \frac{48(3a^2+b^2)}{a^4(5a^2+2b^2)}$$

$$C_{11} = -\frac{2}{3}\,\frac{b^2(6a^2+b^2)}{a^2(5a^2+2b^2)}$$

$$\Psi_{11} = \frac{1}{1+\nu}\,\frac{a^2}{48(3a^2+b^2)}\,[4(1+\nu)^2a^2(5a^2+2b^2) - \nu(1+\nu)\times$$
$$(3a^4 + 6a^2b^2 - b^4) - 4\nu^2b^4]$$

The symbol $[a \leftrightarrow b,\ 1 \leftrightarrow 2]$ denotes the preceding component with $a$ and $b$ and the subscripts 1 and 2 interchanged. Values of $\Gamma_2, A_{22}, B_{22}, C_{22}, \Psi_{22}$ are obtained from the formulas for $\Gamma_1, A_{11}, B_{11}, C_{11}, \Psi_{11}$ by the replacement $a \leftrightarrow b$ and the subscripts $1 \leftrightarrow 2$. The remaining components of the tensors $A^{\alpha\beta}$, $B^{\alpha\beta}$, $C^{\alpha\beta}$, $\Psi^{\alpha\beta}$ are zero on the principal axes of inertia.

4°. Rectangle $|x_1| \leqslant a$, $|x_2| \leqslant b$

$$G = \left\{\frac{x_1}{6}\left[(x_1^2 - 3a^2)\Gamma_1 + \left(\frac{4}{3}x_1^2 + b^2 - a^2\right)\partial_x\chi_1\right] + \right.$$

$$\frac{2b^3}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3\operatorname{ch}(n\pi a/b)}\operatorname{sh}\frac{n\pi x_1}{b}\cos\frac{n\pi x_2}{b}\,\partial_x\chi_1\Big\} + [a\leftrightarrow b,\ 1\leftrightarrow 2] -$$

$$\frac{1}{6}x_\alpha x^\alpha x_\beta \partial_x \chi^\beta$$

$$\Gamma_1 = \frac{5}{6a^2}\left[\left(\frac{b^2}{3}-\frac{2a^2}{5}\right)\partial_x\chi_1 - \frac{9}{a^2}\varkappa_1\right]$$

$$A_{11} = a^4\left(\frac{79}{27\cdot 45} - \frac{a}{b}\frac{4}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{n^5}\operatorname{th}\frac{n\pi b}{a}\right),\quad B_{11} = \frac{15}{a^4},\quad C_{11} = \frac{5b^2}{9a^2}$$

$$\Psi_{11} = \frac{2}{3}a^4\left[\frac{5\nu+6}{15} + \frac{\nu^2}{1+\nu}\left(\frac{25b^4-79a^4}{810a^4} + \frac{a}{b}\frac{2}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{n^5}\operatorname{th}\frac{n\pi b}{a}\right)\right]$$

The values of $\Gamma_2$, $A_{22}$, $B_{22}$, $C_{22}$, $\Psi_{22}$ are obtained by the replacement $a \leftrightarrow b$ and the subscripts $1 \leftrightarrow 2$, while the remaining components of the tensors $A^{\alpha\beta}$, $B^{\alpha\beta}$, $C^{\alpha\beta}$, $\Psi^{\alpha\beta}$ are zero on the principal axes of inertia.

In particular, for a square with side $a$

$$A^{\alpha\beta} = a^4\left(\frac{79}{27\cdot 45} - \frac{4}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{n^5}\operatorname{th} n\pi\right)\delta^{\alpha\beta},\quad B^{\alpha\beta} = \frac{15}{a^4}\delta^{\alpha\beta},\quad C^{\alpha\beta} = \frac{5}{9}\delta^{\alpha\beta}$$

$$\Psi^{\alpha\beta} = \frac{2}{3}a^4\left[\frac{5\nu+6}{15} + \frac{\nu^2}{1+\nu}\left(\frac{2}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{n^5}\operatorname{th} n\pi - \frac{1}{15}\right)\right]\delta^{\alpha\beta}$$

**6. Dispersion equation.** A dispersion equation, the relation between the frequency $\omega$ and the wave number $k$ for which the functions

$$u_\alpha = u_\alpha^\circ e^{i(\omega t - kx)},\qquad u_\alpha^\circ = \mathrm{const}$$

are solutions of the problem of free vibrations of an infinite bar, can be obtained from (3. 1) or (3. 2). Correspondingly, we have for (3. 1) and (3. 2)

$$(\rho\omega^2 - Ek^4I_{11} + \rho\omega^2k^2I_{11} + Ek^6\Psi_{11})(\rho\omega^2 - Ek^4I_{22} + \rho\omega^2k^2I_{22} + Ek^6\Psi_{22}) = (\rho\omega^2k^2I_{12} - Ek^4I_{12} + Ek^6\Psi_{12})^2 \qquad (6.1)$$

$$[\rho\omega^2 - Ek^4I_{11} + \rho\omega^2k^2(I_{11} + \Psi_1^\gamma I_{\gamma_1}^{-1})][\rho\omega^2 - Ek^4I_{22} + \rho\omega^2k^2(I_{22} + \Psi_2^\gamma I_{\gamma_2}^{-1})] = [\rho\omega^2k^2(I_{12} + \Psi_1^\gamma I_{\gamma_2}^{-1}) - Ek^4I_{12}][\rho\omega^2k^2(I_{12} + \Psi_2^\gamma I_{\gamma_1}^{-1}) - Ek^4I_{21}] \qquad (6.2)$$

The dispersion equation is an invariant characteristic of a mechanical system, hence (6. 1) or (6. 2) should yield a second approximation to the exact dispersion equation corresponding to the three-dimensional problem.

As is known, the exact dispersion equation has an infinity of branches. The Bernoulli-Euler equation describes "slow vibrations" and corresponds to the branches $\omega_0(k)$ for which $\omega_0(0) = 0$. Hence, the approximate dispersion equation considered should refine precisely these branches.

An attempt is made in many papers to describe simultaneously two branches of the dispersion curve $\omega_0(k)$ and $\omega_1(k)$, $\omega_1(0) \neq 0$. In this connection, it should be emphasized that the order of smallness is decreased by one for the other branches by differentiation with respect to $x$ and $t$ hence, the problem remains essentially three-dimensional and the meaning of the one-dimensional approximation requires explanation.

The second approximation can be constructed by means of the exact dispersion equation of the three-dimensional problem and the equation in the $x - t$ representation can be reproduced by means of the approximate dispersion equation. Confirmation of the asymptotic accuracy of the approximate equations can be based on this.

Pochhammer [7] constructed the dispersion equation of the three-dimensional problem $f(\omega, k) = 0$ for a bar of circular cross section. Expanding $f(\omega, k)$ in a series in $\omega$, $k$ and keeping terms of the order of $r^2$ and $r^4$, we obtain

$$\rho\omega^2 - \frac{E}{4} r^2k^4 + \rho\,\frac{8\nu + 29}{24}\, r^2\omega^2k^2 - E\,\frac{7\nu + 9}{96\,(1 + \nu)}\, r^4k^6 = 0 \qquad (6.3)$$

After substituting the value of $\Psi_{\alpha\beta}$ for a bar of circular section into (6.1), this latter differs from (6.3) by the term

$$\frac{8\nu + 23}{24}\, r^2k^2\left(\rho\omega^2 - \frac{E}{4} r^2k^4\right)$$

i.e. they are equivalent in the sense mentioned in Sect. 3.

The values of $\Psi_{\alpha\beta}$ calculated in Sect. 5, permit writing a second approximation equation in cases for which the exact equation is unknown.

**7. Asymptotic accuracy of the hypotheses (1.3).** We represent the displacement vector of a point of the bar $w^{i}(x, x^\alpha, t)$ as

$$w^i(x, x^\alpha, t) = u^i(x, t) + e^i_\alpha(x, t)x^\alpha + {}^1/_2\chi^i_{\alpha\beta}(x^\alpha x^\beta - I^{\alpha\beta}) + g^i(x, x^\alpha, t) \qquad i = 0, 1, 2$$

The subscript ° is ordinarily omitted in writing the projections of the vectors and tensors on the $x^\circ$-axis.

Let us require that the functions $g^i$ satisfy the constraints

$$\langle g^i\rangle = 0, \quad \langle g^i x^\alpha\rangle = 0, \quad \langle g^i_{,\alpha} x^\beta\rangle = 0 \qquad (7.1)$$

Then the functions $u^i$, $e^i_\alpha$, $\chi^i_{\alpha\beta}$ will be defined uniquely in terms of the functions $w^i$, namely

$$u^i(x, t) = \langle w^i\rangle, \quad e^i_\alpha(x, t) = I^{-1}_{\alpha\beta}\langle w^i x^\beta\rangle, \quad \chi^i_{\alpha\beta}(x, t) = I^{-1}_{\beta\nu}\langle w^i_{,\alpha} x^\nu\rangle$$

A mutually one-to-one correspondence exists between the set of functions $\{w^i\}$ and a number of sets $\{u^i, e^i_\alpha, \chi^i_{\alpha\beta}, g^i\}$. We obtain the system of equations for the functions $u^i$, $e^i_\alpha$, $\chi^i_{\alpha\beta}$, $g^i$ from the condition for the extremum of the functional (1.2). Hence $g$ is an odd and $g^\alpha$ an even function of $x^\alpha$ in the problem of transverse vibrations. The system of constraints (7.1) reduces to the constraints

$$\langle g^\alpha\rangle = 0, \quad \langle g x^\alpha\rangle = 0, \quad \langle g^\alpha_{,\beta} x^\gamma\rangle = 0 \qquad (7.2)$$

We obtain the following system of Euler equations and boundary conditions in the variation of the functional:

Equations for $u_\alpha$

$$\rho\partial_t^2 u_\alpha - \mu\partial_x(\varphi_\alpha + \langle g_{,\alpha}\rangle) = P_\alpha / \Omega \qquad (7.3)$$

$$\mu(\varphi_\alpha + \langle g_{,\alpha}\rangle) = \mp\langle p_\alpha\rangle, \quad x = \mp l$$

Equations for $e_\alpha$

$$\mu\,(\varphi_\alpha + \langle g_{,\alpha}\rangle) - I^\gamma_\alpha\,(\lambda\partial_x\chi_\gamma + L_2\,(e_\gamma)) = N_\alpha\,/\,\Omega \quad (7.4)$$

$$I^{\alpha\gamma}\,(\lambda\chi_\gamma + (\lambda + 2\mu)\partial_x e_\gamma) = \mp\,\langle px^\alpha\rangle, \quad x = \mp\, l$$

Equations for $\chi^{\alpha\beta\gamma}$

$$2\lambda I^\nu_{(\gamma}\delta_{\beta)\alpha}\,(\chi_\nu + \partial_x e_\nu) + 2\mu\,(\chi_{(\alpha\beta)\nu}\,I^\nu_\gamma + \chi_{(\alpha\gamma)\nu}I^\nu_\beta) - \quad (7.5)$$

$$^1/_2 L_1\,(\chi_{\alpha\sigma\mu})\langle(x^\sigma x^\mu - I^{\sigma\mu})\,(x_\beta x_\gamma - I_{\beta\gamma})\rangle - \langle L_1\,(g_\alpha)\,x_\beta x_\gamma\rangle -$$

$$\mu\langle(x_\beta x_\gamma - I_{\beta\gamma})\partial x g_{,\alpha}\rangle = D_{\alpha\beta\gamma}\,/\,\Omega$$

$$\mu\,[^1/_2\partial_x\chi_{\alpha\mu\sigma}\langle(x^\beta x^\gamma - I^{\beta\gamma})\,(x^\mu x^\sigma - I^{\mu\sigma})\rangle +$$

$$\langle(x^\beta x^\gamma - I^{\beta\gamma})\,(g_{,\alpha} + \partial_x g_\alpha)\rangle] = \mp\,\langle p_\alpha\,(x^\beta x^\gamma - I^{\beta\gamma})\rangle, \quad x = \mp\, l$$

Equations for $g$

$$\mu\Delta g + (\lambda + \mu)\partial_x g^\alpha_{,\alpha} + L_2\,(g) = \mu x^\alpha\gamma_\alpha\,(x,\,t) \quad (7.6)$$

$$\left(g_{,\alpha} + \partial_x g_\alpha + \varphi_\alpha + \frac{1}{2}\,\partial_x\chi_{\alpha\beta\gamma}\,(x^\beta x^\gamma - I^{\beta\gamma})\right)n^\alpha = \frac{1}{\mu}\,p \quad \text{on } \Gamma$$

$$\lambda g^\alpha_{,\alpha} + (\lambda + 2\mu)\partial_x g = \mp\, p + \lambda^\alpha\,(x,\,t)x_\alpha, \quad x = \mp\, l$$

Equations for $g_\alpha$

$$\mu\Delta g_\alpha + (\lambda + \mu)\,\frac{\partial}{\partial x_\alpha}(\partial_x g + g^\beta_{,\beta}) + L_1\,(g_\alpha) + \frac{1}{2}L_1\,(\chi_{\alpha\beta\gamma})\,(x^\beta x^\gamma - I^{\beta\gamma}) = \Lambda_\alpha(x,\,t) \quad (7.7)$$

$$\lambda\,(\partial_x g + g^\beta_{,\beta})n_\alpha + 2\mu g_{(\alpha,\beta)}n^\beta = p_\alpha + \lambda_{\alpha\beta\gamma}x^\beta x^\gamma \quad \text{on } \Gamma$$

$$\mu\,[g_{,\alpha} + \partial_x g_\alpha + \varphi_\alpha + {}^1/_2\partial_x\chi_{\alpha\beta\gamma}\,(x^\beta x^\gamma - I^{\beta\gamma})] = \mp\, p_\alpha, \quad x = \mp\, l$$

Here

$$D_\alpha^{\beta\gamma} = \int_\Gamma p_\alpha\,(x^\beta x^\gamma - I^{\beta\gamma})\,d\Gamma, \quad L_1 = \mu\partial_x^2 - \rho\partial_t^2, \quad L_2 = (\lambda + 2\mu)\,\partial_x^2 - \rho\partial_t^2$$

$$\chi_\alpha = \chi^\beta_{\beta\alpha}$$

$\gamma_\alpha$, $\lambda_\alpha$, $\Lambda_\alpha$, $\lambda_{\alpha\beta\gamma}$ are Lagrange multipliers corresponding to the conditions (7.2).

The system of equations (7.3) — (7.7) is an exact system of equations describing the dynamical bending of a bar of arbitrary thickness. It can be shown that this system is equivalent to the classical equations of three-dimensional elasticity theory. To estimate the orders of the required quantities, we make the change of variable $x^\alpha = hz^\alpha$. Then $0 \leqslant z^\alpha \leqslant 1$.

The system (7.3) — (7.7) can be investigated as a system containing the small parameter $h$ by using formal asymptotic expansions. We hence consider that differentiation with respect to $x$ does not change, but differentiation with respect to $t$ does increase the order of smallness of the quantities by one.

Let us represent the required functions and the given external forces as the following expansions in $h$ ($f$ is any of the functions $u_\alpha$, $e_\alpha$, $\varphi_\alpha$, $\chi_{\alpha\beta\gamma}$, $g$, $g_\alpha$):

$$f = \sum_{s=0}^{\infty} h^s f^{(s)}$$

$$P_\alpha = h^4 \sum_{s=0}^{\infty} h^s P^{(s)}_\alpha, \quad N_\alpha = h^4 \sum_{s=0}^{\infty} h^s N^{(s)}_\alpha, \quad D_{\alpha\beta\gamma} = h^6 \sum_{s=0}^{\infty} h^s D^{(s)}_{\alpha\beta\gamma}$$

$$p_\alpha = h^3 \sum_{s=0}^{\infty} h^s p_\alpha^{(s)}, \quad p = h^2 \sum_{s=0}^{\infty} h^s p^{(s)}$$

The hypothesis about the nature of the differentiation with respect to time corresponds to the fact that all the functions depend on $t$ in terms of the parameter $\tau = ht$. The expansions for the external forces are selected in such a way that the expansions of the functions required could start with $h^0$ but not with negative powers of $h$.

Substituting these expansions into (7.3) — (7.7) and equating coefficients of identical powers of $h$ will yield a system of recursion equations ($\Delta$ is the Laplace operator in the variables $z_\alpha$)

$$\rho\partial_t^2 u_\alpha^{(k-2)} - \mu\partial_x(\varphi_\alpha^{(k)} + \langle g_{,\alpha}^{(k+1)}\rangle) = P_\alpha^{(k-2)} / S \tag{7.8}$$

$$\mu(\varphi_\alpha^{(k)} + \langle g_{,\alpha}^{(k+1)}\rangle) - J^{\alpha\gamma}(\lambda\partial_x\chi_\gamma^{(k-2)} + (\lambda+2\mu)\partial_x^2 e_\gamma^{(k-2)} - \rho\partial_t^2 e_\gamma^{(k-4)}) = N_\alpha^{(k-2)} / S \tag{7.9}$$

$$2\lambda J_{(\gamma}^{\nu}\delta_{\beta)\alpha}(\chi_\nu^{(k)} + \partial_x e_\nu^{(k)}) + 2\mu(\chi_{(\alpha\beta)\nu}^{(k)} J_\gamma^\nu + \chi_{(\alpha\gamma)\nu}^{(k)} J_\beta^\nu) - {}^1/_2(\mu\partial_x^2\chi_{\alpha\sigma\mu}^{(k-2)} - \rho\partial_t^2\chi_{\alpha\sigma\mu}^{(k-4)})\langle(z^\sigma z^\mu - J^{\sigma\mu})(z_\beta z_\gamma - J_{\beta\gamma})\rangle - \langle(\mu\partial_x^2 g_\alpha^{(k)} - \rho\partial_t^2 g_\alpha^{(k-2)}) z_\beta z_\gamma\rangle - \mu\langle\partial_x g_{,\alpha}^{(k+1)}(z_\beta z_\gamma - J_{\beta\gamma})\rangle = D_{\alpha\beta\gamma}^{(k-2)} / S \tag{7.10}$$

$$\mu\Delta g^{(k)} + (\lambda+\mu)\partial_x g_{,\alpha}^{(k-1)\alpha} + (\lambda+2\mu)\partial_x^2 g^{(k-2)} - \rho\partial_t^2 g^{(k-4)} = \mu z^\alpha \gamma_\alpha^{(k-3)} \tag{7.11}$$

$$\mu\Delta g_\alpha^{(k)} + (\lambda+\mu)\frac{\partial}{\partial z_\alpha}(\partial_x g^{(k-1)} + g_{,\beta}^{(k)\beta}) + \mu\partial_x^2 g_\alpha^{(k-2)} - \rho\partial_t^2 g_\alpha^{(k-4)} + {}^1/_2(\mu\partial_x^2\chi_{\alpha\beta\gamma}^{(k-4)} - \rho\partial_t^2\chi_{\alpha\beta\gamma}^{(k-6)})(z^\beta z^\gamma - J^{\beta\gamma}) = \Lambda_\alpha^{(k-2)} \tag{7.12}$$

Here

$$g_{,\alpha} = \partial g / \partial z_\alpha, \quad S = \Omega / h^2, \quad J^{\alpha\beta} = I^{\alpha\beta} / h^2$$

A recursion system of boundary conditions corresponding to (7.8) — (7.12) can be obtained by similar means. Solving (7.11) for $k = 0, 1, 2$ and (7.12) for $k = 0, 1, 2, 3$ successively for the appropriate boundary conditions, we obtain that $\varphi_\alpha^{(2)}$, $g^{(3)}$, $g_\alpha^{(4)}$, respectively, are the first nonzero coefficients in the expansions for $\varphi_\alpha$, $g$, $g_\alpha$. All the quantities with negative ordinal numbers are hence set equal to zero and the solutions of (7.10) for $\chi_{\alpha\beta\gamma}$ for $k = 0, 1$ are used.

Let us solve (7.10) in a first approximation. It is easy to confirm that any third-rank tensor which is symmetric in the last two subscripts can be represented as (the symmetrization operation is marked by parentheses)

$$\chi_{\alpha\beta\gamma} = \chi_{(\alpha\beta)\gamma} + \chi_{(\alpha\gamma)\beta} - \chi_{(\beta\gamma)\alpha} \tag{7.13}$$

In a first approximation (for $k = 0$) (7.10) is of the form

$$2\lambda J^{\nu(\gamma}\delta^{\beta)\alpha}(\chi_\nu^{(0)} + \partial_x e_\nu^{(0)}) + 2\mu(\chi^{(0)(\alpha\beta)}{}_\nu J^{\gamma\nu} + \chi^{(0)(\alpha\gamma)}{}_\nu J^{\beta\nu}) = 0 \tag{7.14}$$

Performing a convolution of (7.14) with $\delta^{\alpha\beta}$ and $\delta^{\beta\gamma}$ in sequence, we obtain after calculations

$$\chi_\alpha^{(0)} = -\frac{\lambda}{\lambda+\mu}\partial_x e_\alpha^{(0)} \tag{7.15}$$

$$\chi_{(\alpha\beta)\gamma}^{(0)} = {}^1/_2\,\delta_{\alpha\beta}\chi_\gamma^{(0)} \tag{7.16}$$

From (7.13) and (7.16) we have

$$\chi^{(0)}_{\alpha\beta\gamma} = {}^1/_2\, (\chi^{(0)}_{\gamma}\delta_{\alpha\beta} + \chi^{(0)}_{\beta}\delta_{\alpha\gamma} - \chi^{(0)}_{\alpha}\delta_{\beta\gamma}) \qquad (7.17)$$

For $k = 1$ Eq. (7.10) agrees with (7.14), and therefore, its solution is given by (7.17) and (7.15).

In a first approximation, we obtain the Bernoulli-Euler equation from (7.8), (7.9) and (7.15). We therefore have for the orders of the required quantities

$$\varphi_\alpha \sim h^2, \quad g \sim h^3, \quad g_\alpha \sim h^4$$

After substituting the displacements in the form

$$w = e_\alpha x^\alpha + g, \quad w_\alpha = u_\alpha + {}^1/_2\chi_{\alpha\beta\gamma}\,(x^\beta x^\gamma - I^{\beta\gamma}) + g_\alpha$$

it can be seen from the formula for the functional (1.2) that the integrand contains terms of the order of $h^2$, $h^4$, $h^6$, where the components containing $g_\alpha$ are of the order of $h^6$ and should be omitted in deriving the second approximation equations. Therefore, to obtain the second approximation it is necessary to set $g_\alpha = 0$. Computing $\chi_{\alpha\beta\gamma}$ by means of (7.17), we obtain the following asymptotically exact representation of the displacements: $w = e_\alpha x^\alpha + g$, $w_\alpha = u_\alpha + {}^1/_4\chi_\beta R^\beta_\alpha$.

The case when differentiation with respect to time does not alter the order of smallness of the quantities is considered similarly and results in the same representation for the displacements.

**8. Comparison with other results from refined theories.** To obtain one-dimensional equations of the theory of bars, the method of hypotheses and the method of series expansions in the transverse coordinates are usually applied.

The classical Bernoulli-Euler beam vibration equation is based on the hypothesis for the displacements

$$w = -\partial_x u_\alpha\,(x, t)x^\alpha, \quad w_\alpha = u_\alpha\,(x, t) + {}^1/_4\chi_\beta\,(x, t)R_\alpha{}^\beta \qquad (8.1)$$

Substituting (8.1) into (1.2), assuming that differentiation with respect to $t$ raises the order of smallness by one, and discarding terms of order $h^4$ and $h^6$ in the integrand, we arrive at the functional

$$I = \Omega\int_{t_1}^{t_2} dt\int_{-l}^{l}\left(\frac{1}{2}\rho\partial_t u_\alpha\partial_t u^\alpha - F\right)dx + \int_{t_1}^{t_2} dt\int_{-l}^{l}(u_\alpha P^\alpha - \partial_x u_\alpha N^\alpha)\,dx + \int_{t_1}^{t_2} dt\,\Big[u_\alpha\langle p^\alpha\rangle - \partial_x u_\alpha\langle px^\alpha\rangle\Big]_{-l}^{l} \qquad (8.2)$$

$$F = {}^1/_2 I^{\alpha\beta}\,[(\lambda + \mu)\chi_\alpha\chi_\beta - 2\lambda\chi_\alpha\partial_x{}^2u_\beta + (\lambda + 2\mu)\partial_x{}^2u_\alpha\partial_x{}^2u_\beta]$$

The functional (8.2) differs from the Bernoulli-Euler functional in the form of the function $F$. In order to obtain the known expression, we note that the parameters $\chi_\alpha$ enter into the functional (8.2) without derivatives, hence, the Euler equations for $\chi_\alpha$ are algebraic and easily solved

$$\chi_\alpha = \frac{\lambda}{\lambda+\mu}\,\partial_x{}^2u_\alpha$$

Substituting the value of $\chi_\alpha$ into the expression for the elastic energy of unit length $F$ results in the formula

$$F = {}^1/_2 EI^{\alpha\beta}\partial_x{}^2u_\alpha\partial_x{}^2u_\beta$$

The Rayleigh equation [12] is substantially based also on the hypothesis (8.1) and corresponds to the fact that the component $\rho I^{\alpha\beta}\partial_t\partial_x u_\alpha\partial_t\partial_x u_\beta$ of the order of $h^4$ and describing the rotational inertia of the cross section is kept in the functional (8.2). It follows from the above that the Rayleigh model is not asymptotically exact.

The Timoshenko equation [13] is based on the representation of the displacement (1.3) which is valid for the second approximation. However, hypotheses reducing to two equalities

$$\langle \sigma_\alpha^\alpha x^\beta \rangle = 0, \quad q_\alpha = k\mu\Omega\varphi_\alpha \tag{8.3}$$

$$(\sigma^{\alpha\beta} = \partial U / \partial \varepsilon_{\alpha\beta}, \quad \sigma_\alpha = \partial U / \partial \varepsilon_\alpha, \quad q_\alpha = \Omega\langle\sigma_\alpha\rangle)$$

($q_\alpha$ are the transverse force components and $k$ is the shear factor) are used in place of the solutions of the equations written above for the functions $\chi_\alpha$ and $g$. From (8.3) follow

$$\chi_\alpha = -\frac{\lambda}{\lambda+\mu}\partial_x e_\alpha, \quad \varphi_\alpha + \langle g_{,\alpha}\rangle = k\varphi_\alpha \tag{8.4}$$

respectively, and the system of equations for the functions $u_\alpha$ and $e_\alpha$ turns out to be closed. Solving this system for $u_\alpha$, we obtain the Timoshenko beam equation. In the free vibrations case it has the form

$$\rho\partial_t^2 u_\alpha + EI_\alpha^\beta \partial_x^4 u_\beta - \rho I_\alpha^\beta\left(1+\frac{E}{\mu k}\right)\partial_t^2\partial_x^2 u_\beta + \frac{\rho^2}{\mu k} I_\alpha^\beta \partial_t^4 u_\beta = 0 \tag{8.5}$$

It follows from the results of Sects. 2 and 7 that the relationships (8.4) are not correct in the second approximation. The shear coefficient $k$ in the Timoshenko model is considered dependent on only the shape of the cross section. It should be emphasized that if $k$ is introduced by using the second formula of (8.4), $k = q_1 / (\mu\Omega\varphi_1)$ (it is understood that such a definition is possible if the vibrations occur in the $(x, x_1)$-plane) and is evaluated by the asymptotically exact theory, then $k$ will depend not only on the shape of the cross section, but also on the external load (even for $p = 0$).

Another method of defining the coefficient $k$ can be proposed. Let us discard the small last term in a second approximation in the Timoshenko equation (8.5). Then (3.2) and (8.5) will be identical in form. For a complete agreement between the equations describing the vibrations along the $x_1$-axis (in cases admitting of separation of the vibrations), $k$ should be defined by the formula

$$k = \frac{E}{\mu}\frac{(I_{11})^2}{\Psi_{11}} \tag{8.6}$$

A considerable quantity of papers (see the survey [17]) is devoted to evaluation of the shear coefficient $k$ for different cross sections. To evaluate $k$ Timoshenko used the stress distribution in a section according to elementary theory. A solution of the Saint-Venant problem about a cantilever beam was used in [22] to evaluate $k$ by means of the second formula in (8.4). The equations in [22] have been obtained also by using the hypotheses (8.3) and are therefore not asymptotically exact.

For the cases of a circle, ellipse and rectangle, the formula (8.6) yields, respectively,

$$k^{-1} = \frac{4\nu^2 + 12\nu + 7}{6(1+\nu)^2} \quad \text{(circle)} \tag{8.7}$$

$$k^{-1} = \frac{4(1+\nu)^2(5+2m^{-2}) - 4\nu^2 m^{-4} - \nu(1+\nu)(3+6m^{-2}-m^{-4})}{6(1+\nu)^2(3+m^{-2})} \quad \text{(ellipse)} \tag{8.8}$$

$$k^{-1} = \frac{191\nu^2 + 594\nu + 324}{270(1+\nu)^2} + \frac{\nu^2}{(1+\nu)^2}\cdot\left(\frac{5}{54}m^{-4} + m\frac{6}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{n^5}\operatorname{th} n\pi m^{-1}\right) \quad \text{(rectangle)} \tag{8.9}$$

Here $m = a / b$, the semi-axis $a$ is along the $x_1$-axis. For vibrations along the $x_2$-axis, $m$ should be replaced by $m^{-1}$ in (8.8) and (8.9).

Presented below are numerical values of the coefficient $k$ for a circle and rectangle for $\nu = 0.3$, from the data of different authors

| | [13] | [22] | [14] | Formulas (8.7), (8.9) |
|---|---|---|---|---|
| Circle | 0.750 | 0.866 | 0.614 | 0.930 |
| Rectangle | 0.833 | 0.850 | — | 0.872 ($m = 1$) |

We note that in the case of a rectangle $k$ is independent of the ratio between the sides $m$ according to Timoshenko [13] and Cowper [22]. It is seen from (8.9) that this assumption results in large errors in the evaluation of $k$ corresponding to vibrations along different sides of the rectangle. For example, the values of $k$ differ by 35% even for $m = 3$ (at $\nu = 0.3$).

It should be emphasized that defining $k$ by (8.6) permits only the attainment of agreement of the resolving equations. The system of second approximation equations (2.7) — (2.9) and the system of Timoshenko equations generally remain different.

The approximate equations of transverse vibrations of a circular cross section bar were obtained in [14] from the three-dimensional dynamic equations by using a formal symbolic method. As has been shown in [23], the approximate equations of [14] are not asymptotically exact and do not agree with the Pochhammer equations. We note that the method in [14] is only suitable for a circular bar.

Expansion of the displacement vector into a double power series in the transverse coordinates and integrating the dynamic equations of three-dimensional elasticity theory over the bar section reduce the initial problem to an infinite system of one-dimensional differential equations. Jacobi or Legendre polynomials are sometimes used in the expansions. Different approximate theories are obtained by cutting off the infinite series. This method was used in bar theory mainly to derive the longitudinal vibrations equations. The second approximation equations for transverse vibrations were obtained in [15, 16] by the series expansion method. Methods of cutting off the series mentioned in these papers do not yield asymptotically exact equations. It should also be emphasized that the displacement vector in a second approximation cannot be represented as a polynomial in a finite power of the transverse coordinates for every cross section. For example, according to Sect. 5, the displacement vector in a second approximation is not a polynomial for a rectangular or annular section.

**9. Appendix.** Separation of transverse and longitudinally torsional vibrations in bars with centrally-symmetric cross section. Let us consider an anisotropic, linearly-elastic rectilinear bar with constant centrally-symmetric cross section. There is a plane of elastic symmetry perpendicular to the axis at each point of the bar. The bar is generally inhomogeneous, however its elastic properties are centrally-symmetric. We show that the transverse and longitudinal-torsional vibrations are independent in such a bar.

The concept of evenness can be introduced for functions defined in centrally-symmetric domains. Each function is represented as the sum of an even and odd function (the even functions are later denoted by a double, and the odd by a single prime).

Let us consider the appropriate representation for the displacements

$$w(x^\alpha, x, t) = w'(x^\alpha, x, t) + w''(x^\alpha, x, t),$$

$$w_\alpha(x^\beta, x, t) = w_\alpha'(x^\beta, x, t) + w_\alpha''(x^\beta, x, t)$$

The arguments $x$, $t$ enter as parameters into these representations. Differentiation with respect to $x$ and $t$ does not change the evenness, but does it with respect to $x^\alpha$.

Because of substituting the representations mentioned into the functional (1.2), this latter is separated into the sum of two functionals, one of which depends only on $w_\alpha''$, $w'$, and the other on $w_\alpha'$, $w''$. In fact, the strain tensor components $\varepsilon_{\alpha\beta}'$, $\varepsilon'$, $\varepsilon_\alpha''$ depend on $w_\alpha''$, $w'$, and the components $\varepsilon_{\alpha\beta}''$, $\varepsilon''$, $\varepsilon_\alpha'$ on $w_\alpha'$, $w''$. There follows from the central symmetry of the elastic properties that the components of the elastic modulus tensor are even functions of $x^\alpha$ and from the existence of a plane of elastic symmetry that there are no products $\varepsilon_{\alpha\beta}\varepsilon^\alpha$ and $\varepsilon\varepsilon^\alpha$ in the elastic energy. Hence, the integral of the elastic energy over the cross section is separated into a sum of two integrals, one of which contains $\varepsilon_{\alpha\beta}'$, $\varepsilon'$ and $\varepsilon_\alpha''$ and the other $\varepsilon_{\alpha\beta}''$, $\varepsilon''$, $\varepsilon_\alpha'$. Similarly, the integral of the kinetic energy over the cross section can be separated into integrals dependent on $\partial_t w_\alpha''$, $\partial_t w'$ and $\partial_t w_\alpha'$, $\partial_t w''$. Because of the linearity, the functional describing the work of the external forces is separated correspondingly.

The variational problem under consideration is separated completely into two independent problems since the even and odd components of the functions can be varied independently. It is natural to call the problem containing the even component of $w_\alpha$ and the odd component of $w$ the problem of transverse vibrations and the problem containing the odd component of $w_\alpha$ and the even component of $w$ the longitudinal-torsional vibrations problem.

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## EQUILIBRIUM OF A SLOPE WITH A TECTONIC CRACK

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Equilibrium of an elastic half-plane with a rectilinear crack reaching the half-plane free boundary at an arbitrary angle is considered as a plane problem of the theory of elasticity. It is assumed that known compressive stresses are applied at considerable distance from the crack forcing the opposite boundaries of the crack to contact each other. Interaction between the crack boundaries are defined by the law of dry friction with cohesion. Mathematically this problem is analogous to that of a tectonic crack filled with a low-strength medium. First, the problem is stated and fundamental relationships are presented. The Wiener-Hopf equation of the considered problem is derived with the use of Mellin transform and Jones method. The exact analytical solution of the Wiener-Hopf equations is then